Analytical solutions for ln, sqrt, asin, asinh, acos, acosh, atan, atanh on branch cuts using minus zero

Anton Shterenlikht

Mech Eng Dept, The University of Bristol, Bristol BS8 1TR, UK

In the following $z = x + iy$. DLMF is NIST Digital Library of Mathematical Functions, `http://dlmf.nist.gov`.

The polar form, for $z \neq 0$:

$$z = |z| \exp \mathrm{Arg} z$$

where Arg z is (DLMF, Sec 1.9(i), Eqns. 1.9.5, 1.9.6):

quadrant	x	y	Arg z
1st	$\geq 0, +0$	$\geq 0, +0$	ω
2nd	$\leq 0, -0$	$\geq 0, +0$	$\pi - \omega$
3rd	$\leq 0, -0$	$\leq 0, -0$	$-\pi + \omega$
4th	$\geq 0, +0$	$\leq 0, -0$	$-\omega$

where

$$\omega = \mathrm{atan} \left| \frac{y}{x} \right| \qquad \omega \in [0, \pi/2]$$

1. LOG

From DLMF (Sec 4.2(i), Eqn. 4.2.3):

$$\ln z = \ln |z| + i \mathrm{Arg} z$$

log has a single branch cut along the negative real axis (DLMF Sec 4.2(i), Fig. 4.2.1). We need to examine the top and the bottom boundaries of the cut, for $|z| < -1$, $|z| = -1$ and for $|z| > -1$. So 6 special values must be examined.

1.1. $z = -a - i0, a > 0$

This value is in the third quadrant, with $\omega = 0$, hence

$$\mathrm{Arg} z = -\pi$$

If $-\infty < a \leq -1$ then $|z| \geq 1$, therefore $\ln |z| \geq 0$.

If $-1 \leq a < 0$ then $|z| \leq 1$, therefore $\ln |z| \leq 0$.

If $a = 1$ then $|z| = 1$, therefore $\ln |z| = 0$.

Finally

$$\ln z = b - i\pi$$

where $b \geq 0$ for $a \geq 1$, and $b \leq 0$ for $0 < a \leq 1$.

1.2. $z = -a + i0, a > 0$

This value is in the second quadrant, with $\omega = 0$, hence

$$\mathrm{Arg} z = +\pi$$

the rest is as in the previous case. Finally

$$\ln z = b + i\pi$$

where $b \geq 0$ for $a \geq 1$, and $b \leq 0$ for $0 < a \leq 1$.

2. SQRT

If $z = |z|(\cos\theta + i\sin\theta)$ then

$$\sqrt{z} = \sqrt{|z|}\left(\cos\frac{\theta + 2n\pi}{2} + i\sin\frac{\theta + 2n\pi}{2}\right) = \sqrt{|z|}\left(\cos(\frac{\theta}{2} + n\pi) + i\sin(\frac{\theta}{2} + n\pi)\right)$$

$\sqrt{z}$ has a single branch cut: $x \leq 0$, $y = 0$, or $\theta = \pm\pi$. Hence $\sqrt{z}$ on the branch cut is on the positive imaginary axis, $+\frac{\pi}{2}$, for $\theta = +\pi$, and on the negative imaginary axis, $-\frac{\pi}{2}$ for $\theta = -\pi$.

In particular:

$$\sqrt{0 + i0} = 0 + i0$$

$$\sqrt{0 - i0} = 0 - i0$$

3. ASIN

From DLMF (Sec 4.23(iv), Eqn. 4.23.19):

$$\text{asin} z = -i\ln(\sqrt{1 - z^2} + iz)$$

asin has 2 branch cuts. We need to examine the top and the bottom boundaries of both cuts. So 4 special values must be examined.

3.1. $z = -a - i0, a \geq 1$

$$iz = i(-a - i0) = 0 - ia$$

$$z^2 = (-a - i0)(-a - i0) = a^2 - 0 + i0 + i0 = a^2 + i0$$

$$1 - z^2 = 1 - a^2 - i0$$

$$\sqrt{1 - z^2} = \sqrt{1 - a^2 - i0}$$

Now, the argument of $\sqrt{\ }$ is a complex number with a non-positive real part and a *negative* imaginary part. Therefore $\sqrt{1 - a^2 - i0}$ lies on the negative imaginary axis. We can express it like this:

$$\sqrt{1 - z^2} = 0 - i\sqrt{a^2 - 1}$$

$$\sqrt{1 - z^2} + iz = 0 - i\sqrt{a^2 - 1} + 0 - ia = 0 - i(\sqrt{a^2 - 1} + a)$$

Remember that $\ln z = \ln|z| + i\text{Arg}z$.

The imaginary part of the last expression was ≤ 0, so it lies in 4th quadrant, with $\omega = \frac{\pi}{2}$. Therefore

$$\text{Arg}(\sqrt{1 - z^2} + iz) = -\frac{\pi}{2}$$

$$\ln(\sqrt{1 - z^2} + iz) = \ln|\sqrt{a^2 - 1} + a| - i\frac{\pi}{2}$$

Clearly the expression in || is real and $\geq a$, and since $a \geq 1$, it is ≥ 1. Hence instead of || one can simply write $(\sqrt{a^2 - 1} + a) \geq 1$ and therefore $\ln(\sqrt{a^2 - 1} + a) \geq 0$.

Finally

$$-i\ln(\sqrt{1-z^2}+iz) = -i\ln(\sqrt{a^2-1}+a) - \frac{\pi}{2} = -\frac{\pi}{2} - ib$$

where $b = \ln(\sqrt{a^2-1}+a) \geq 0$.

Hence

$$\text{asin}z = -\frac{\pi}{2} - ib,\ b \geq 0$$

This value lies in the 3rd quadrant.

3.2. $z = -a + i0,\ a \geq 1$

$$iz = i(-a+i0) = -0 - ia$$

$$z^2 = (-a+i0)(-a+i0) = a^2 - 0 - i0 - i0 = a^2 - i0$$

$$1 - z^2 = 1 - a^2 + i0$$

$$\sqrt{1-z^2} = \sqrt{1-a^2+i0}$$

Now, the argument of $\sqrt{\ }$ is a complex number with a non-positive real part and a *positive* imaginary part. Even though the imaginary part is zero, it is still positive. Therefore $\sqrt{1-a^2+i0}$ lies on the positive imaginary axis. We can express it like this:

$$\sqrt{1-z^2} = 0 + i\sqrt{a^2-1}$$

$$\sqrt{1-z^2} + iz = 0 + i\sqrt{a^2-1} - 0 - ia = 0 + i(\sqrt{a^2-1} - a)$$

When $a = 1$ then $\sqrt{a^2-1} - a = -1$. When $a \to +\infty$ then $\sqrt{a^2-1} - a \to -0$. So

$$\sqrt{a^2-1} - a \leq 0$$

Hence

$$|\sqrt{a^2-1} - a| = a - \sqrt{a^2-1}$$

Remember that $\ln z = \ln|z| + \text{Arg}z$. From above

$$\text{Arg}(\sqrt{1-z^2}+iz) = -\frac{\pi}{2}$$

$$0 < a - \sqrt{a^2-1} \leq 1$$

So

$$\ln(a - \sqrt{a^2-1}) \leq 0$$

and

$$\ln(\sqrt{1-z^2}+iz) = \ln(a - \sqrt{a^2-1}) - i\,\frac{\pi}{2}$$

Finally

$$-i\ln(\sqrt{1-z^2}+iz) = -i\ln(a - \sqrt{a^2-1}) - \frac{\pi}{2} = -\frac{\pi}{2} + ib$$

where

$$b = -\ln(a - \sqrt{a^2-1}) = \ln(\sqrt{a^2-1}+a) \geq 0$$

Hence

$$\text{asin} z = -\frac{\pi}{2} + ib, \, b \geq 0$$

This value lies in the 2nd quadrant.

3.3. $z = a - i0, a \geq 1$

$$iz = i(a - i0) = 0 + ia$$

$$z^2 = (a - i0)(a - i0) = a^2 - 0 - i0 - i0 = a^2 - i0$$

$$1 - z^2 = 1 - a^2 + i0$$

$$\sqrt{1 - z^2} = \sqrt{1 - a^2 + i0}$$

Now, the argument of $\sqrt{\ }$ is a complex number with a non-positive real part and a *positive* imaginary part. Therefore $\sqrt{1 - a^2 + i0}$ lies on the positive imaginary axis. We can express it like this:

$$\sqrt{1 - z^2} = 0 + i\sqrt{a^2 - 1}$$

$$\sqrt{1 - z^2} + iz = 0 + i\sqrt{a^2 - 1} + 0 + ia = 0 + i(\sqrt{a^2 - 1} + a)$$

Remember that $\ln z = \ln |z| + i\text{Arg} z$.

The imaginary part of the last expression is ≥ 0, so

$$\text{Arg}(\sqrt{1 - z^2} + iz) = \frac{\pi}{2}$$

$$\ln(\sqrt{1 - z^2} + iz) = \ln |\sqrt{a^2 - 1} + a| + i\,\frac{\pi}{2}$$

The expression in $|\ldots|$ is ≥ 0, so the above can be rewritten as

$$\ln(\sqrt{1 - z^2} + iz) = \ln(\sqrt{a^2 - 1} + a) + i\,\frac{\pi}{2}$$

and since $a \geq 1$, then $\ln(\sqrt{a^2 - 1} + a) \geq 0$.

Finally

$$-i\ln(\sqrt{1 - z^2} + iz) = -i\ln(\sqrt{a^2 - 1} + a) + \frac{\pi}{2} = \frac{\pi}{2} - ib$$

where $b = \ln(\sqrt{a^2 - 1} + a) \geq 0$.

Hence

$$\text{asin} z = \frac{\pi}{2} - ib, \, b \geq 0$$

This value lies in the 4th quadrant.

3.4. $z = a + i0, a \geq 1$

$$iz = i(a + i0) = -0 + ia$$

$$z^2 = (a + i0)(a + i0) = a^2 - 0 + i0 + i0 = a^2 + i0$$

$$1 - z^2 = 1 - a^2 - i0$$

$$\sqrt{1 - z^2} = \sqrt{1 - a^2 - i0}$$

Now, the argument of $\sqrt{\ }$ is a complex number with a non-positive real part and a *negative* imaginary part.

Even though the imaginary part is zero, it is still negative. Therefore $\sqrt{1 - a^2 - i0}$ lies on the negative imaginary axis. We can express it like this:

$$\sqrt{1 - z^2} = 0 - i\sqrt{a^2 - 1}$$

$$\sqrt{1 - z^2} + iz = 0 - i\sqrt{a^2 - 1} - 0 + ia = 0 + i(a - \sqrt{a^2 - 1})$$

When $a = 1$ then $a - \sqrt{a^2 - 1} = 1$. When $a \to +\infty$ then $a - \sqrt{a^2 - 1} \to +0$. The imaginary part is ≥ 0. The magnitude is $0 < a - \sqrt{a^2 - 1} \leq 1$.

Remember that $\ln z = \ln |z| + \mathrm{Arg} z$. Therefore from above

$$\mathrm{Arg}(\sqrt{1 - z^2} + iz) = \frac{\pi}{2}$$

$$\ln(\sqrt{1 - z^2} + iz) = \ln(a - \sqrt{a^2 - 1}) + i\,\frac{\pi}{2}$$

Finally

$$-i\ln(\sqrt{1 - z^2} + iz) = -i\ln(a - \sqrt{a^2 - 1}) + \frac{\pi}{2} = \frac{\pi}{2} + ib$$

where

$$b = -\ln(a - \sqrt{a^2 - 1}) = \ln(\sqrt{a^2 - 1} + a) \geq 0$$

Hence

$$\mathrm{asin} z = \frac{\pi}{2} + ib, b \geq 0$$

This value lies in the 1st quadrant.

Finally, looking at the 4 special cases, one can confirm that

$$-\mathrm{asin}(z) = \mathrm{asin}(-z)$$

3.5. Summary

In summary

z	asin z
$-a - i0$	$-\pi/2 - ib$
$-a + i0$	$-\pi/2 + ib$
$a - i0$	$\pi/2 - ib$
$a + i0$	$\pi/2 + ib$

where in all cases $b = \ln(\sqrt{a^2 - 1} + a) \geq 0$.

From the table above it's easy to verify the identity (DLMF 4.23(iii), Eqn. 4.23.10):

$$\mathrm{asin}(-z) = -\mathrm{asin}(z)$$

4. ASINH

From DLMF (Sec 4.37(iv), Eqn. 4.37.16):

$$\mathrm{asinh} z = \ln(\sqrt{1 + z^2} + z)$$

whereas

$$\mathrm{asin} z = -i\ln(\sqrt{1 - z^2} + iz)$$

So asinh can be expressed via asin as:

$$\operatorname{asinh} z = i \operatorname{asin}(-iz)$$

asinh has 2 branch cuts, along the imaginary axis - from i to $+i\infty$, and from $-i$ to $-i\infty$. We need to examine the left and right boundaries of both cuts. So 4 special values must be examined.

4.1. $z = -0 - ia, a \geq 1$

$$iz = i(-0 - ia) = a - i0$$

$$-iz = -a + i0$$

From the table in the previous section:

$$\operatorname{asin}(-a + i0) = -\frac{\pi}{2} + ib$$

Finally

$$\operatorname{asinh} z = i(-\frac{\pi}{2} + ib) = -b - i\frac{\pi}{2}$$

where $b \geq 0$.

4.2. $z = 0 - ia, a \geq 1$

$$iz = i(0 - ia) = a + i0$$

$$-iz = -a - i0$$

From the table in the previous section:

$$\operatorname{asin}(-a - i0) = -\frac{\pi}{2} - ib$$

Finally

$$\operatorname{asinh} z = i(-\frac{\pi}{2} - ib) = b - i\frac{\pi}{2}$$

where $b \geq 0$.

4.3. $z = -0 + ia, a \geq 1$

$$iz = i(-0 + ia) = -a - i0$$

$$-iz = a + i0$$

From the table in the previous section:

$$\operatorname{asin}(a + i0) = \frac{\pi}{2} + ib$$

Finally

$$\operatorname{asinh} z = i(\frac{\pi}{2} + ib) = -b + i\frac{\pi}{2}$$

where $b \geq 0$.

4.4. $z = 0 + ia, a \geq 1$

$$iz = i(0 + ia) = -a + i0$$

$$-iz = a - i0$$

From the table in the previous section:

$$\operatorname{asin}(a - i0) = \frac{\pi}{2} - ib$$

Finally

$$\operatorname{asinh} z = i(\frac{\pi}{2} - ib) = b + i\,\frac{\pi}{2}$$

where $b \geq 0$.

4.5. Summary

In summary

z	asinh z
$+0 + ia$	$b + i\pi/2$
$-0 + ia$	$-b + i\pi/2$
$+0 - ia$	$b - i\pi/2$
$-0 - ia$	$-b - i\pi/2$

where in all cases $b = \ln(\sqrt{a^2 - 1} + a) \geq 0$.

From the table above it's easy to verify the identity (DLMF 4.37(iii), Eqn. 4.37.10):

$$\operatorname{asinh}(-z) = -\operatorname{asinh}(z)$$

5. ACOS

From DLMF (Sec 4.23(iv), Eqns. 4.23.19 and 4.23.22):

$$\operatorname{acos} z = \frac{\pi}{2} - \operatorname{asin} z$$

This is not a definition, but rather a convenient mathematical identity. Since we have already studied asin, this identity is the easiest way to study behaviour of acos on the branch cuts.

acos has the same 2 branch cuts as asin. So we use the same 4 values from the cuts.

5.1. $z = -a - i0, a \geq 1$

We already showed that for this z

$$\operatorname{asin} z = -\frac{\pi}{2} - ib, b \geq 0$$

Hence

$$\operatorname{acos} z = \frac{\pi}{2} + \frac{\pi}{2} + ib = \pi + ib, b \geq 0$$

This value lies in the 1st quadrant.

5.2. $z = -a + i0, a \geq 1$

We already showed that for this z

$$\operatorname{asin} z = -\frac{\pi}{2} + ib, b \geq 0$$

Hence

$$\operatorname{acos} z = \frac{\pi}{2} + \frac{\pi}{2} - ib = \pi - ib, b \geq 0$$

This value lies in the 4th quadrant.

5.3. $z = a - i0, a \geq 1$

We already showed that for this z

$$\operatorname{asin} z = \frac{\pi}{2} - ib, b \geq 0$$

Hence

$$\text{acos} z = \frac{\pi}{2} - \frac{\pi}{2} + ib = 0 + ib, b \geq 0$$

This value lies on the positive imaginary axis.

5.4. $z = a + i0, a \geq 1$

We already showed that for this z

$$\text{asin} z = \frac{\pi}{2} + ib, b \geq 0$$

Hence

$$\text{acos} z = \frac{\pi}{2} - \frac{\pi}{2} - ib = 0 - ib, b \geq 0$$

This value lies on the negative imaginary axis.

5.5. Summary

In summary

z	acos z
$-a - i0$	$\pi + ib$
$-a + i0$	$\pi - ib$
$a - i0$	$0 + ib$
$a + i0$	$0 - ib$

where in all cases $b = \ln(\sqrt{a^2 - 1} + a) \geq 0$.

From the table above it's easy to verify the identity (DLMF 4.23(iii), Eqn. 4.23.11):

$$\text{acos}(-z) = \pi - \text{acos}(z)$$

6. ACOSH

From DLMF (Sec 4.37(iv), Eqn. 4.37.21:

$$\text{acosh} z = 2\ln\left(\sqrt{\frac{z+1}{2}} + \sqrt{\frac{z-1}{2}}\right)$$

acosh has a single branch cut along the real axis from 1 to $-\infty$.

6.1. $z = a + i0, a \leq -1$

$$\sqrt{\frac{z+1}{2}} = \sqrt{\frac{a+1+i0}{2}}$$

$$\sqrt{\frac{z-1}{2}} = \sqrt{\frac{a-1+i0}{2}}$$

The real parts of both expressions under $\sqrt{\ }$ are ≤ 0. The imaginary parts of both expressions under $\sqrt{\ }$ are $+0$, meaning that the Arg of both expressions under $\sqrt{\ }$ are $+\pi$. Hence the principal values of both square roots are on the positive imaginary axis:

$$\sqrt{\frac{z+1}{2}} = 0 + i\sqrt{\frac{-a-1}{2}}$$

$$\sqrt{\frac{z-1}{2}} = 0 + i\sqrt{\frac{-a+1}{2}}$$

$$\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}=0+i\left(\sqrt{\frac{-a-1}{2}}+\sqrt{\frac{-a+1}{2}}\right)$$

The expression in () is ≥ 1, so ln() ≥ 0:

$$\ln\left(\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right)=\ln\left(\sqrt{\frac{-a-1}{2}}+\sqrt{\frac{-a+1}{2}}\right)+i\,\frac{\pi}{2}$$

Finally

$$\text{acosh}z=2\ln\left(\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right)=b+i\pi$$

where

$$b=2\ln\left(\sqrt{\frac{-a-1}{2}}+\sqrt{\frac{-a+1}{2}}\right)\geq 0$$

6.2. $z=a+i0, -1\leq a\leq 1$

$$\sqrt{\frac{z+1}{2}}=\sqrt{\frac{a+1+i0}{2}}$$

The real part of the expression under $\sqrt{\ }$ is ≥ 0. Hence:

$$\sqrt{\frac{z+1}{2}}=\sqrt{\frac{a+1}{2}}$$

However in

$$\sqrt{\frac{z-1}{2}}=\sqrt{\frac{a-1+i0}{2}}$$

the real part of the expression under $\sqrt{\ }$ are ≤ 0. The imaginary part of the expression under $\sqrt{\ }$ is +0, meaning that the Arg of the expression under $\sqrt{\ }$ is $+\pi$. Hence the principal value is on the positive imaginary axis:

$$\sqrt{\frac{z-1}{2}}=0+i\sqrt{\frac{-a+1}{2}}$$

$$\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}=\sqrt{\frac{a+1}{2}}+i\sqrt{\frac{-a+1}{2}}$$

The absolute value of this complex number is:

$$\left|\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right|=\left(\frac{a+1}{2}+\frac{-a+1}{2}\right)^{1/2}=1$$

and the Arg is:

$$\text{Arg}\left(\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right)=\text{atan}\sqrt{\frac{-a+1}{a+1}}$$

As $a\rightarrow -1$ Arg $\rightarrow +\frac{\pi}{2}$. As $a\rightarrow 1$ Arg $\rightarrow 0$. If $a=0$ then Arg $=\frac{\pi}{4}$. So

$$\ln\left(\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right)=0+i\operatorname{atan}\sqrt{\frac{-a+1}{a+1}}$$

Finally

$$\operatorname{acosh}z=0+i2\operatorname{atan}\sqrt{\frac{-a+1}{a+1}}$$

As $a\to-1$ Im $\to+\pi$. As $a\to1$ Im $\to0$. If $a=0$ then Im $=+\frac{\pi}{2}$.

6.3. $z=a-i0,-1\le a\le1$

$$\sqrt{\frac{z+1}{2}}=\sqrt{\frac{a+1-i0}{2}}$$

The real part of the expression under $\sqrt{\ }$ is ≥0. Hence:

$$\sqrt{\frac{z+1}{2}}=\sqrt{\frac{a+1}{2}}$$

However in

$$\sqrt{\frac{z-1}{2}}=\sqrt{\frac{a-1-i0}{2}}$$

the real part of the expression under $\sqrt{\ }$ are ≤0. The imaginary part of the expression under $\sqrt{\ }$ is -0, meaning that the Arg of the expression under $\sqrt{\ }$ is $-\pi$. Hence the principal value is on the negative imaginary axis:

$$\sqrt{\frac{z-1}{2}}=0-i\sqrt{\frac{-a+1}{2}}$$

$$\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}=\sqrt{\frac{a+1}{2}}-i\sqrt{\frac{-a+1}{2}}$$

The absolute value of this complex number is:

$$\left|\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right|=\left(\frac{a+1}{2}+\frac{-a+1}{2}\right)^{1/2}=1$$

and the Arg is:

$$\operatorname{Arg}\left(\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right)=\operatorname{atan}\left(-\sqrt{\frac{-a+1}{a+1}}\right)=-\operatorname{atan}\sqrt{\frac{-a+1}{a+1}}$$

As $a\to-1$ Arg $\to-\frac{\pi}{2}$. As $a\to1$ Arg $\to0$. If $a=0$ then Arg $=-\frac{\pi}{4}$. So

$$\ln\left(\sqrt{\frac{z+1}{2}}+\sqrt{\frac{z-1}{2}}\right)=0-i\operatorname{atan}\sqrt{\frac{-a+1}{a+1}}$$

Finally

$$\operatorname{acosh}z=0-i2\operatorname{atan}\sqrt{\frac{-a+1}{a+1}}$$

As $a\to-1$ Im $\to-\pi$. As $a\to1$ Im $\to0$. If $a=0$ then Im $=-\frac{\pi}{2}$.

6.4. $z = a - i0, a \le -1$

$$\sqrt{\frac{z+1}{2}} = \sqrt{\frac{a+1-i0}{2}}$$

$$\sqrt{\frac{z-1}{2}} = \sqrt{\frac{a-1-i0}{2}}$$

The real parts of both expressions under $\sqrt{\ }$ are ≤ 0. The imaginary parts of both expressions under $\sqrt{\ }$ are -0, meaning that the Arg of both expressions under $\sqrt{\ }$ are $-\pi$. Hence the principal values of both square roots are on the negative imaginary axis:

$$\sqrt{\frac{z+1}{2}} = 0 - i\sqrt{\frac{-a-1}{2}}$$

$$\sqrt{\frac{z-1}{2}} = 0 - i\sqrt{\frac{-a+1}{2}}$$

$$\sqrt{\frac{z+1}{2}} + \sqrt{\frac{z-1}{2}} = 0 - i\left(\sqrt{\frac{-a-1}{2}} + \sqrt{\frac{-a+1}{2}}\right)$$

The expression in () is ≥ 1, so $\ln() \ge 0$:

$$\ln\left(\sqrt{\frac{z+1}{2}} + \sqrt{\frac{z-1}{2}}\right) = \ln\left(\sqrt{\frac{-a-1}{2}} + \sqrt{\frac{-a+1}{2}}\right) - i\,\frac{\pi}{2}$$

Finally

$$\text{acosh}z = 2\ln\left(\sqrt{\frac{z+1}{2}} + \sqrt{\frac{z-1}{2}}\right) = b - i\pi$$

where

$$b = 2\ln\left(\sqrt{\frac{-a-1}{2}} + \sqrt{\frac{-a+1}{2}}\right) \ge 0$$

6.5. Summary

In summary

z	a	acosh z
$a + i0$	$a \le -1$	$b + i\pi$
$a + i0$	$-1 \le a \le +1$	$0 + ic$
$a - i0$	$-1 \le a \le +1$	$0 - ic$
$a - i0$	$a \le -1$	$b - i\pi$

where in all cases

$$b = 2\ln\left(\sqrt{\frac{-a-1}{2}} + \sqrt{\frac{-a+1}{2}}\right) \ge 0$$

$$c = 2\text{atan}\sqrt{\frac{-a+1}{a+1}}$$

As $a \to -1$: $c \to +\pi$. As $a \to 1$: $c \to 0$. When $a = 0$: $c = \frac{\pi}{2}$.

From the table above it's easy to verify the identity (DLMF 4.37(iii), Eqn. 4.37.11):

$$\operatorname{acosh}(-z) = \pm i\pi + \operatorname{acosh}(z)$$

for $-1 \le a \le 1$.

7. ATAN

On can use expression in DLMF Sec. 4.23(iv), Eqn. 4.23.27 for the cuts. Atan has 2 branch cuts along the imaginary axis, from $+i$ to $+\infty$, and from $-i$ to $-\infty$, DLMF Sec. 4.23(ii), Fig. 4.23.1(ii). Note that DLMF expression for atan in Sec. 4.23(iv), Eqn. 4.23.26 has branch cuts along the real axis, which is why it cannot be used on atan branch cuts. Here's why.

DLMF Eqn. 4.23.26 is:

$$\operatorname{atan} z = \frac{i}{2} \ln \frac{i+z}{i-z}$$

The expression for atan that can be derived from atanh (Kahan, 1987):

$$\operatorname{atan} z = \operatorname{atanh}(iz)/i$$

which leads to:

$$\operatorname{atan} z = \frac{i}{2} \ln \frac{1-iz}{1+iz}$$

While the two atan expressions are seemingly identical:

$$\frac{i}{2} \ln \frac{i+z}{i-z} = \frac{i}{2} \ln \frac{-i(i+z)}{-i(i-z)} = \frac{i}{2} \ln \frac{1-iz}{1+iz}$$

multiplication by i under ln moves the branch cut from the real to the imaginary axis. It is easy to show that on the branch cut

$$\ln(iz) \ne \ln i + \ln z$$

So in this section we use this expression for atan:

$$\operatorname{atan} z = \frac{i}{2} \ln \frac{1-iz}{1+iz} = \frac{i}{2} (\ln(1-iz) - \ln(1+iz))$$

which is valid everywhere in C, including the branch cuts.

We need to examine the left and the right boundaries of both cuts. So 4 special values must be examined.

7.1. The 1st quadrant: $z = +0 + ia,\ a \ge 1$

$$iz = i(+0 + ia) = +i0 - a$$

$$1 - iz = 1 - i0 + a = a + 1 - i0$$

$$\ln(1 - iz) = \ln(a + 1 - i0) = \ln(a+1) - i0$$

because the value is in the 4th quadrant and $\operatorname{atan}\left|\frac{-0}{a+1}\right| = 0$.

$$1 + iz = 1 + i0 - a = 1 - a + i0$$

$$\ln(1 + iz) = \ln(1 - a + i0) = \ln(a - 1) + i\pi$$

because the value is in the 2nd quadrant and $\operatorname{atan}\left|\frac{0}{1-a}\right| = 0$.

$$\ln(1 - iz) - \ln(1 + iz) = \ln(a + 1) - i0 - \ln(a - 1) - i\pi = \ln \frac{a + 1}{a - 1} - i\pi$$

Finally

$$\text{atanz} = \frac{i}{2} (\ln \frac{a + 1}{a - 1} - i\pi) = \frac{i}{2} \ln \frac{a + 1}{a - 1} + \frac{\pi}{2} = \frac{\pi}{2} + \frac{i}{2} \ln \frac{a + 1}{a - 1}$$

where if $a \to 1$ then Im $\to +\infty$ and when $a \to +\infty$ then Im $\to +0$.

7.2. The 2nd quadrant: $z = -0 + ia, a \geq 1$

$$iz = i(-0 + ia) = -i0 - a$$

$$1 - iz = 1 + i0 + a = a + 1 + i0$$

$$\ln(1 - iz) = \ln(a + 1 + i0) = \ln(a + 1) + i0$$

because the value is in the 1st quadrant and atan$\left| \frac{0}{a + 1} \right| = 0$.

$$1 + iz = 1 - i0 - a = 1 - a - i0$$

$$\ln(1 + iz) = \ln(1 - a - i0) = \ln(a - 1) - i\pi$$

because the value is in the 3rd quadrant and atan$\left| \frac{-0}{1 - a} \right| = 0$.

$$\ln(1 - iz) - \ln(1 + iz) = \ln(a + 1) + i0 - \ln(a - 1) + i\pi = \ln \frac{a + 1}{a - 1} + i\pi$$

Finally

$$\text{atanz} = \frac{i}{2} (\ln \frac{a + 1}{a - 1} + i\pi) = \frac{i}{2} \ln \frac{a + 1}{a - 1} - \frac{\pi}{2} = -\frac{\pi}{2} + \frac{i}{2} \ln \frac{a + 1}{a - 1}$$

where if $a \to 1$ then Im $\to +\infty$ and when $a \to +\infty$ then Im $\to +0$.

7.3. The 3rd quadrant: $z = -0 - ia, a \geq 1$

$$iz = i(-0 - ia) = -i0 + a$$

$$1 - iz = 1 + i0 - a = 1 - a + i0$$

$$\ln(1 - iz) = \ln(1 - a + i0) = \ln(a - 1) + i\pi$$

because the value is in the 2nd quadrant and atan$\left| \frac{-0}{1 - a} \right| = 0$.

$$1 + iz = 1 - i0 + a = 1 + a - i0$$

$$\ln(1 + iz) = \ln(1 + a - i0) = \ln(a + 1) - i0$$

because the value is in the 4th quadrant and atan$\left| \frac{-0}{1 + a} \right| = 0$.

$$\ln(1 - iz) - \ln(1 + iz) = \ln(a - 1) + i\pi - \ln(a + 1) + i0 = \ln \frac{a - 1}{a + 1} + i\pi$$

Finally

$$\text{atan}z = \frac{i}{2}(\ln\frac{a-1}{a+1} + i\pi) = \frac{i}{2}\ln\frac{a-1}{a+1} - \frac{\pi}{2} = -\frac{\pi}{2} - \frac{i}{2}\ln\frac{a+1}{a-1}$$

where if $a \to 1$ then Im $\to -\infty$ and when $a \to +\infty$ then Im $\to -0$.

7.4. The 4th quadrant: $z = +0 - ia, a \geq 1$

$$iz = i(+0 - ia) = +i0 + a$$

$$1 - iz = 1 - i0 - a = 1 - a - i0$$

$$\ln(1 - iz) = \ln(1 - a - i0) = \ln(a - 1) - i\pi$$

because the value is in the 3rd quadrant and atan$\left|\frac{-0}{1-a}\right| = 0$.

$$1 + iz = 1 + i0 + a = 1 + a + i0$$

$$\ln(1 + iz) = \ln(1 + a + i0) = \ln(a + 1) + i0$$

because the value is in the 1st quadrant and atan$\left|\frac{0}{a+1}\right| = 0$.

$$\ln(1 - iz) - \ln(1 + iz) = \ln(a - 1) - i\pi - \ln(a + 1) - i0 = \ln\frac{a-1}{a+1} - i\pi$$

Finally

$$\text{atan}z = \frac{i}{2}(\ln\frac{a-1}{a+1} - i\pi) = \frac{i}{2}\ln\frac{a-1}{a+1} + \frac{\pi}{2} = \frac{\pi}{2} - \frac{i}{2}\ln\frac{a+1}{a-1}$$

where if $a \to 1$ then Im $\to -\infty$ and when $a \to +\infty$ then Im $\to -0$.

7.5. Summary

In summary

quadrant	z	atan z
1	$+0 + ia$	$+\pi/2 + ib$
2	$-0 + ia$	$-\pi/2 + ib$
3	$-0 - ia$	$-\pi/2 - ib$
4	$+0 - ia$	$+\pi/2 - ib$

where in all cases

$$b = \frac{1}{2}\ln\frac{a+1}{a-1} \geq 0$$

From the table above it's easy to verify the identity (DLMF 4.23(iii), Eqn. 4.23.12):

$$\text{atan}(-z) = -\text{atan}(z)$$

8. ATANH

From DLMF 4.37(iv), Eqn. 4.37.24:

$$\text{atanh}z = \frac{1}{2}\ln\frac{1+z}{1-z}$$

However, since we have calculated special values for atan$z$ already, it's easier to use this identity (Kahan, 1987):

$$\text{atan}z = \text{atanh}(iz)/i$$

or

$$\operatorname{atan}(-iz) = \frac{1}{i}\operatorname{atanh} z$$

or

$$i\operatorname{atan}(-iz) = \operatorname{atanh} z$$

atanh has 2 branch cuts along the real axis, from −1 to −∞ and from +1 to +∞.

8.1. The 1st quadrant: $z = +a + i0,\ a \geq 1$

$$-iz = -i(+a + i0) = -ia + 0 = +0 - ia$$

From the previous section, $\operatorname{atan}(+0 - ia) = +\frac{\pi}{2} - \frac{i}{2}\ln\frac{a+1}{a-1}$, so

$$\operatorname{atanh} z = i\left(+\frac{\pi}{2} - \frac{i}{2}\ln\frac{a+1}{a-1}\right) = i\,\frac{\pi}{2} + \frac{1}{2}\ln\frac{a+1}{a-1} = \frac{1}{2}\ln\frac{a+1}{a-1} + i\,\frac{\pi}{2}$$

8.2. The 2nd quadrant: $z = -a + i0,\ a \geq 1$

$$-iz = -i(-a + i0) = +ia + 0 = +0 + ia$$

From the previous section, $\operatorname{atan}(+0 + ia) = +\frac{\pi}{2} + \frac{i}{2}\ln\frac{a+1}{a-1}$, so

$$\operatorname{atanh} z = i\left(+\frac{\pi}{2} + \frac{i}{2}\ln\frac{a+1}{a-1}\right) = i\,\frac{\pi}{2} - \frac{1}{2}\ln\frac{a+1}{a-1} = -\frac{1}{2}\ln\frac{a+1}{a-1} + i\,\frac{\pi}{2}$$

8.3. The 3rd quadrant: $z = -a - i0,\ a \geq 1$

$$-iz = -i(-a - i0) = +ia - 0 = -0 + ia$$

From the previous section, $\operatorname{atan}(-0 + ia) = -\frac{\pi}{2} + \frac{i}{2}\ln\frac{a+1}{a-1}$, so

$$\operatorname{atanh} z = i\left(-\frac{\pi}{2} + \frac{i}{2}\ln\frac{a+1}{a-1}\right) = -i\,\frac{\pi}{2} - \frac{1}{2}\ln\frac{a+1}{a-1} = -\frac{1}{2}\ln\frac{a+1}{a-1} - i\,\frac{\pi}{2}$$

8.4. The 4th quadrant: $z = +a - i0,\ a \geq 1$

$$-iz = -i(+a - i0) = -ia - 0 = -0 - ia$$

From the previous section, $\operatorname{atan}(-0 + ia) = -\frac{\pi}{2} - \frac{i}{2}\ln\frac{a+1}{a-1}$, so

$$\operatorname{atanh} z = i\left(-\frac{\pi}{2} - \frac{i}{2}\ln\frac{a+1}{a-1}\right) = -i\,\frac{\pi}{2} + \frac{1}{2}\ln\frac{a+1}{a-1} = +\frac{1}{2}\ln\frac{a+1}{a-1} - i\,\frac{\pi}{2}$$

8.5. Summary

In summary

quadrant	z	atanh z
1	$+a + i0$	$+b + i\pi/2$
2	$-a + i0$	$-b + i\pi/2$
3	$-a - i0$	$-b - i\pi/2$
4	$+a - i0$	$+b - i\pi/2$

where in all cases

$$b = \frac{1}{2} \ln \frac{a+1}{a-1} \geq 0$$

From the table above it’s easy to verify the identity (DLMF 4.37(iii), Eqn. 4.37.12):

$$\mathrm{atanh}(-z) = -\mathrm{atanh}(z)$$

References

Kahan, 1987.

W. Kahan, “Branch Cuts for Complex Elementary Functions or Much Ado About Nothing’s Sign Bit” in *The State of the Art in Numerical Analysis*, ed. A. Iserles and M. J. D. Powell, Clarendon Press, Oxford (1987).